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2003 J. Phys. A: Math. Gen. 36 12453

(http://iopscience.iop.org/0305-4470/36/50/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 12453–12473 PII: S0305-4470(03)63082-9

Graded contractions of Jordan algebras and of their representations

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Received 1 May 2003
Published 2 December 2003
Online at stacks.iop.org/JPhysA/36/12453

Abstract

Contractions of Jordan algebras and Jordan superalgebras which preserve a chosen grading are defined and studied. Simultaneous grading of Jordan algebras and their representation spaces is used to develop a theory of grading, preserving contractions of representations of Jordan algebras.

PACS numbers: 02.10.De, 02.20.Bb

1. Introduction

The purpose of this paper is to introduce graded contractions of Jordan algebras and their representations the same way as was done for Lie algebras and their representations [1, 2]. The possibility of doing that stems from the possibility of simultaneously grading the algebras and their representation spaces by automorphisms of finite order.

Jordan algebras emerged in physics [3] in order to axiomatize the algebraic relations of quantum-mechanical observables. During the 20th century the theory was further developed mainly by mathematicians. Expectations that Jordan theory could provide the algebraic foundations of quantum theory did not fully materialize: the only Jordan structures totally unrelated to associative structures are finite dimensional. Nevertheless, operator algebras are often used in quantum (statistical) mechanics [7]. In [5], Jordan algebras were applied to complex analysis to give an algebraic description of the bounded or unbounded domains of holomorphy in $\mathbb{C}^n$ arising in the theory of automorphic functions. Recently, quite unexpected connections were found between Jordan algebras (Jordan triple systems) and KdV equations [4]. It seems that representations of Jordan algebras are connected in this case with the linear spectral problem. Interesting applications of Jordan algebras are also in [8, 9]. Further exposition, addressed particularly to the physicist, can be found in [6]. For the Jordan superalgebra see [17].

As for Lie algebras, graded contractions of Jordan algebras are degenerations which preserve a fixed grading of the algebra. Such a grading is obtained as a decomposition of the algebra into the direct sum of eigenspaces of a chosen automorphism of the algebra. The same automorphism also acts on any representation space. Their simultaneous grading thus amounts to the decomposition of the algebra and of its representations into eigenspaces of the same automorphism.

In the traditional approach, the contraction of an algebra is the continuous limit of a parametrized family of isomorphic algebras. There is a considerable body of literature on this subject [11, 12], as well as on the more general notion of the deformation of an algebra [13]. It also harbours the main obstacles to a satisfactory theory: First, in its full generality the study of all continuous deformations of the structure constants of an algebra offers such a bewildering array of possibilities that one cannot hope to obtain precise and practical information except in the simplest cases. Second, to be widely useful, deformations of algebras need to be accompanied by deformations of their representations. The traditional theory offers little guidance in that respect.

The process of graded contraction is functorial and depends on the grading group G rather than on details of the structure of the algebra itself. Moreover, for any fixed grading semigroup the problem is solved simultaneously for Jordan algebras and Jordan superalgebras. By considering, along with the graded algebras, their compatibly graded representations, we also obtain a theory of contractions of representations.

First we recall the definition of both Jordan algebras and superalgebras, some of their properties and illustrate them by basic examples. We introduce (bi)-representations (or (bi)modules) of Jordan algebras following [16]. Then we also define the notion of a special representation for Jordan algebra which seems to be rather popular among both mathematicians and physicists. Since the backbone of our method lies in the study of gradings, we do not require other facts about Jordan algebras. However, further information about them may be found in [15, 10] and, in detail, in [14].

In section 3, we bring up some results from [18, 19] about gradings of Jordan algebras. First, we show that groups of automorphisms of Jordan algebra can be used to exploit the properties of their elements of finite order. They allow us to find many non-equivalent gradings of the algebras and their representations. Second, we formulate some results for gradings in the case of simple Jordan algebra of bilinear form.

In section 4, the graded contractions of Jordan algebras (be it finite or infinite dimensional) are introduced together with the graded contractions of Jordan superalgebras.

In section 5, we will illustrate our method by applying it to an arbitrary Jordan algebra or superalgebra with $\mathbb{Z}_2$-, $\mathbb{Z}_3$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings.

In section 6, the general method of contractions of representations of Jordan algebras is given. The representation theory of recently classified Jordan superalgebras is rather incomplete at present. Therefore such representations are not considered here.

In section 7, one finds other useful examples of contractions of simultaneously graded Jordan algebras and their representations.

Finally, in section 8, we demonstrate methods of comparison for the contraction of the tensor product of two special Jordan modules with the tensor product of their contractions.

2. Jordan algebras and their representations

We restrict our considerations to algebras over the complex number field $\mathbb{C}$. Recall that a $\mathbb{C}$-vector space J together with a composition $J \times J \to J, (x, y) \to x \cdot y$, is called a

Jordan algebra, if all $x, y, z \in J$ and all $\alpha \in \mathbb{C}$ satisfy

$$\begin{aligned} &x \cdot y = y \cdot x \qquad \text{(commutativity)} \\ &x \cdot (y + z) = x \cdot y + x \cdot z \qquad (\alpha x) \cdot y = \alpha(xy) \\ &(x^2 \cdot y)x = x^2(y \cdot x) \qquad \text{where} \quad x^2 = x \cdot x \qquad \text{(Jordan identity)}. \end{aligned} \tag{1}$$

The Jordan identity can be rewritten in terms of the *associator* (x, y, z),

$$(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$$

as $(x, y, x^2) = 0$. Furthermore, replacing x by $x + \lambda z$, $\lambda \in \mathbb{C}$ and comparing the coefficients of λ, we get the so-called *polarization formula*:

$$2(x, y, z \cdot x) + (z, y, x^2) = 0 \qquad \text{for all} \quad x, y, z \in J.$$

Replacing x by $x + \lambda w$ in the last formula gives us the *multilinear* identity

$$(x, y, z \cdot w) + (w, y, z \cdot x) + (z, y, x \cdot w) = 0 \qquad \text{for all} \quad x, y, z, w \in J. \tag{2}$$

Subsequently, the generalized Jordan identity (2) is recalled whenever we need to check whether an algebra is Jordan.

Let A be an associative algebra. In analogy with the commutator $[x, y]$ we may introduce a symmetric (Jordan) multiplication

$$x \cdot y = \tfrac{1}{2}(xy + yx) \qquad x, y \in A.$$

The algebra obtained after introducing the multiplication $x \cdot y$ on the vector space A is denoted by $A^{(+)}$. Then $A^{(+)}$ is a Jordan algebra and every subspace in A closed with respect to the operation $x \cdot y = \frac{1}{2}(xy + yx)$, forms a subalgebra of $A^{(+)}$ and is consequently a Jordan algebra. Such an algebra is called a *special* Jordan algebra.

Let V be a vector space of dimension greater than 1 over $\mathbb{C}$, with a symmetric non-degenerate bilinear form $f(x, y)$. Let us consider the direct sum of vector spaces $J(V, f) = \mathbb{C} \cdot 1 \oplus V$. We define multiplication in the following way:

$$(\alpha + x) * (\beta + y) = \alpha\beta + f(x, y) \cdot 1 + (\alpha y + \beta x) \qquad \text{for all} \quad \alpha, \beta \in \mathbb{C} \quad x, y \in V.$$

Then $(J(V, f), *)$ is a *simple special Jordan algebra*.

Let G be an Abelian finite group and suppose J is a Jordan algebra graded by G. That means we have the grading decomposition

$$J = \bigoplus_{i \in G} J_i$$

of J into the direct sum of grading subspaces J_i, such that for every choice of $x \in J_i$ and $y \in J_j$ we have

$$x \cdot y = z$$

where z belongs to the grading subspace J_{i+j}. As a shorthand we rewrite this relation in terms of the multiplication of the subspaces

$$J_i \cdot J_j \subseteq J_{i+j} \qquad i, j, i + j \in G. \tag{3}$$

Here we are using the additive notation for the multiplication in G.

Any $\mathbb{Z}_2$-graded algebra $J = J_0 \oplus J_1$ is called a superalgebra. A superalgebra J is said to be commutative if

$$x \cdot y = (-1)^{\alpha\beta} y \cdot x \qquad \text{for} \quad x \in J_\alpha \quad y \in J_\beta. \tag{4}$$

As was defined in [17], a Jordan superalgbera is a commutative superalgebra J with an operation $\cdot$ which satisfies the following axiom:

$$(-1)^{\alpha\gamma}(x, y, z \cdot w) + (-1)^{\beta\alpha}(w, y, z \cdot x) + (-1)^{\gamma\beta}(z, y, x \cdot w) = 0 \tag{5}$$

where $x \in J_\alpha$, $z \in J_\beta$, $w \in J_\gamma$, $y \in J$. We will call the last equation a super-Jordan identity.

The basic question of representation theory is about the problem of embedding a Jordan algebra into an associative algebra A. We define such embeddings to be a homomorphism of J into Jordan algebra $A^{(+)}$ obtained by replacing ordinary multiplication in an associative algebra by Jordan multiplication. Of particular interest are the embeddings in matrix algebras, or what amounts to the same thing, in algebra of linear transformations $\mathrm{End}(V)$, where V is a vector space. More precisely, if J is a Jordan algebra we call a vector space V with bilinear compositions $J \times V \to V$, where $(x, m) \to xm$ and $V \times J \to V$, where $(m, x) \to mx$ a Jordan *bimodule* if the split algebra

$$J \oplus V = (J \oplus V, \star) \qquad (x, m) \star (y, n) = x \cdot y + xn + my$$

for all $x, y, z \in J$ and $m, n \in V$, is again a Jordan algebra. In terms of elements, we get from (1) and (2)

$$\begin{aligned} &mx = xm \\ &(x \cdot y, m, z) + (y \cdot z, m, x) + (z \cdot x, m, y) = 0 \\ &(x \cdot z, y, m) + (m \cdot x, y, z) + (m \cdot z, y, x) = 0. \end{aligned} \tag{6}$$

Consequently, a *birepresentation* is defined by two embeddings $L_x : J \to \mathrm{End}(V)$, sending $m \to xm$ and $R_x : J \to \mathrm{End}(V)$, sending $m \to mx$.

It is clear that since $mx = xm$, a Jordan bimodule can also be considered as a Jordan right or left module. In the following the term Jordan module will be used for the left Jordan module.

In particular, any vector space V together with a bilinear map $J \times V \to V$, such that

$$(x \cdot y)m = x(ym) + y(xm) \qquad \text{for any} \quad x, y \in J \quad m \in V \tag{7}$$

defines a Jordan module for J. A module of this type will be called *special*. For reference see [8, 10].

Suppose that V and W are two special modules for a Jordan algebra J and let $U = V \otimes W$. Then it is clear that we can define compositions $U \times J \to U$ and $J \times U \to U$ by setting

$$(m_1 \otimes m_2)x = m_1 \otimes xm_2 \qquad x(m_1 \otimes m_2) = xm_1 \otimes m_2$$

$m_1 \in V$, $m_2 \in W$ and $x \in J$. Finally, we consider the new linear map $\{ , \} : U \times J \to U$,

$$\{x, (m_1 \otimes m_2)\} = xm_1 \otimes m_2 + m_1 \otimes xm_2 \tag{8}$$

which defines a Jordan module in U, see [16]. Let us stress here that the new module structure we got on U has a structure of the Jordan (bi)module (not special Jordan module).

3. Gradings of simple Jordan algebras

In this section we want to provide a sufficiently large selection of gradings, so that subsequent consideration of graded contractions is not deprived of explicit examples. In no way do we attempt here to exhaust all the possible gradings.

Various gradings of a Jordan algebra are obtained by means of decomposition of Jordan algebra into eigenspaces of one or several commuting automorphisms of the algebra. Two gradings are equivalent if they can be transformed into each other by means of an automorphism of the algebra. Examples of non-equivalent automorphisms providing equivalent gradings of the algebra are abundant.

In general, non-equivalent gradings of finite-dimensional Jordan algebras have apparently not been classified. A major step towards that goal, of interest on its own, would be a description of all fine gradings (i.e. gradings which cannot be further refined) of simple Jordan algebras.

It is quite likely that interesting objects related to such gradings would be their stabilizers, particularly the subgroup of the stabilizer which permutes the subspaces of the decomposition. See [21, 22] for curious examples of this kind from the Lie theory.

Conjugacy classes of elements of finite order of compact simply connected Lie groups are known [20], and their description in any representation is given in [18]. Therefore they can be directly used for the grading of Jordan subalgebras of $M_n(\mathbb{C})^{(+)}$.

Let us consider the Jordan algebra $M_n(\mathbb{C})^{(+)}$ of square matrices $n \times n$. As the grading automorphisms, we consider elements of finite order, $g^N = 1$, in $SU(n)$. The grading is accomplished once we know the matrices X in

$$gXg^{-1} = \lambda X \tag{9}$$

for all eigenvalues λ. Thus it suffices to know each g only in the defining representation of $SU(n)$, that is as a unitary matrix $n \times n$. From each conjugacy class of such automorphisms, it is convenient to choose its unique element represented by a diagonal matrix.

Let us now describe the diagonal matrices representing each conjugacy class. An $SU(n)$-conjugacy class of elements of finite order is completely characterized by n integers $[s_0, s_1, \ldots, s_{n-1}]$ such that

$$\{s_0, s_1, \ldots, s_{n-1}\} \in \mathbb{Z}^{\geqslant 0} \qquad \text{and} \qquad \gcd\{s_0, s_1, \ldots, s_{n-1}\} = 1.$$

Then the eigenvalues λ in (9) are the Mth roots of unity, where $M = \sum_{i=0}^{n-1} s_i$. It is useful to visualize s_i as attached to the nodes of the extended Dynkin diagram of type A_{n-1}.

Finally following section 9, remark 4 of [18], let us write an explicit expression for the diagonal matrices g in three cases:

$$n = 2: \quad g = \begin{pmatrix} \exp\left(\pi \mathrm{i} \frac{s_1}{s_0+s_1}\right) & 0 \\ 0 & \exp\left(\pi \mathrm{i} \frac{-s_1}{s_0+s_1}\right) \quad 0 \end{pmatrix} \tag{10}$$

$$n = 3: \quad g = \begin{pmatrix} \exp\left(\frac{2\pi \mathrm{i}}{3} \frac{2s_1+s_2}{s_0+s_1+s_2}\right) & 0 & 0 \\ 0 & \exp\left(\frac{2\pi \mathrm{i}}{3} \frac{-s_1+s_2}{s_0+s_1+s_2}\right) & 0 \\ 0 & 0 & \exp\left(\frac{2\pi \mathrm{i}}{3} \frac{-s_1-2s_2}{s_0+s_1+s_2}\right) \end{pmatrix} \tag{11}$$

$$n \geqslant 2: \quad g = \begin{pmatrix} \exp\left(\frac{2\pi \mathrm{i}}{nM} U_1\right) & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \exp\left(\frac{2\pi \mathrm{i}}{nM} U_k\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \exp\left(\frac{2\pi \mathrm{i}}{nM} U_{n-1}\right) \end{pmatrix} \tag{12}$$

$$1 \leqslant k \leqslant n \qquad M = \sum_{j=0}^{n-1} s_j \qquad U_k = -\sum_{q=0}^{k-1} q s_q + \sum_{p=k}^{n} (n-p) s_p.$$

Note that the adjoint action (9) of g on the algebra is of order M, but the element g is of order nM/c, where $c = \gcd\{n, s_1, \ldots, s_{n-1}\}$.

As an example of special cases of the diagonal matrices above, one can verify that the elements of the centre of $SU(n)$, which are all multiples of the identity matrix, have $M = 1$ and that there are precisely n of them all distinct. Indeed, their conjugacy classes are given by $[s_0, s_1, \ldots, s_{n-1}] = [1, 0, \ldots, 0], [0, 1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$.

To illustrate, let us consider a simple special Jordan algebra

$$J = M_3(\mathbb{C})^{(+)} = \{X \mid X \in \mathbb{C}^{3\times 3}\} = \begin{pmatrix} a & d & g \\ h & b & e \\ f & j & c \end{pmatrix} \tag{13}$$

graded by the element $g = \mathrm{diag}\{\theta, 0, \theta^2\}$, where θ is a root of unity. Then the adjoint action of g on $X \in M_3(\mathbb{C})$ gives

$$gXg^{-1} = \begin{pmatrix} a & \theta d & \theta^{-1} g \\ \theta^{-1} h & b & \theta^{-2} e \\ \theta f & \theta^2 j & c \end{pmatrix}.$$

Suppose that $\theta^2 = 1$, then we immediately have $\mathbb{Z}_2$-grading on J

$$J = J_0 \oplus J_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & e \\ 0 & j & c \end{pmatrix} \oplus \begin{pmatrix} 0 & d & g \\ h & 0 & 0 \\ f & 0 & 0 \end{pmatrix}. \tag{14}$$

By applying simultaneously the adjoint action by the element $h = \mathrm{diag}\{1, \gamma, \gamma^2\}$ we obtain $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on J

$$\begin{aligned} J &= J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11} \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & j & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ f & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & d & 0 \\ h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

Let us remark that both of the gradings that we have obtained, give also the gradings for Jordan algebras of symmetric matrices $M = M^T$.

Suppose that $\theta^3 = 1$ (in particular, $\theta = \theta^{-2}$). Then we get a $\mathbb{Z}_3$-grading for J

$$J = J_0 \oplus J_1 \oplus J_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \oplus \begin{pmatrix} 0 & d & 0 \\ 0 & 0 & e \\ f & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & g \\ h & 0 & 0 \\ 0 & j & 0 \end{pmatrix}. \tag{16}$$

Let us now bring up some results about gradings of simple Jordan algebras of bilinear form $J(V, f)$. Due to [19], any grading $J(V, f) = \bigoplus_{i \in G} J_i$ by a group G over a field (char $k \neq 2$) can be described as follows. There exists a graded basis B of V, which is a disjoint union of $B = B_1 \cup B_1' \cup F$ and a bijection $B \ni b \leftrightarrow b' \in B'$ such that $\deg b = (\deg b')^{-1} = 1$ for any $b \in B$ and $(\deg f)^2 = 1$ for any $f \in F$. The sets B, and B' are dual to each other, the duality is established by the same bijection. F is orthonormal and orthogonal to both B, B'.

As an example let us consider a three-dimensional vector space V and a symmetric bilinear form f_1 on V such that in some basis $\langle v_1, v_2, v_3 \rangle$ we have $f_1(v_1, v_2) = f_1(v_3, v_3) = 1$ and for any other pair (i, j) $f_1(v_i, v_j) = 0$. Then we have a $\mathbb{Z}_3$-grading of

$$J = J_0 \oplus J_1 \oplus J_2 = \langle 1, v_3 \rangle \oplus \langle v_1 \rangle \oplus \langle v_2 \rangle. \tag{17}$$

4. Graded contractions of Jordan algebras and superalgebras

In this section we will describe the method which allows us to find the contractions which preserve a chosen fixed grading of a Jordan algebra or superalgebra. There is no restriction as to the dimension of the algebra, it may be finite or infinite. No features are required other than the presence of the chosen grading. We continually speak about Jordan algebras only occasionally underlining the fact that superalgebras with the same type of grading are being considered as well. Let G be an Abelian group and J is a Jordan algebra graded by G.

Let us now define the G-graded contractions. Suppose a G-graded Jordan algebra J is given. A G-graded contraction J^ε of J:

$$J \to J^\varepsilon = \bigoplus_{i \in G} J_i$$

is the Jordan algebra J^ε with the grading decomposition isomorphic to that of J and with the contracted multiplications:

$$J_i \cdot^\varepsilon J_j = \varepsilon_{ij} J_i \cdot J_j \subseteq \varepsilon_{ij} J_{i+j} \tag{18}$$

given by the contraction parameters $\varepsilon_{ij} \in \mathbb{C}$. Here by $J_i \cdot J_j$ we mean a linear space generated by the products of every element of J_i with every element of J_j. On the element level we have $x \cdot^\varepsilon y = \varepsilon_{ij} z$, where $x \in J_i$, $y \in J_j$, $z \in J_{i+j}$ and $x \cdot y = z$.

The special case of contractions arises when the multiplication relations (18) are modified by renormalization of the basis of the subspaces by arbitrary non-zero constants,

$$J_i \to a_i J_i \qquad i \in G \quad a_i \in \mathbb{C} \quad a_i \neq 0.$$

Then in terms of the new basis we will have

$$J_i \cdot J_j \subseteq \frac{a_i a_j}{a_{i+j}} J_{i+j} \qquad i, j \in G.$$

Obviously such a $\varepsilon = (\varepsilon_{ij}) = \left(\frac{a_i a_j}{a_{i+j}}\right)$ determines a contraction. We will call two contractions ε_1 and ε_2 equivalent if one can be obtained from the other by renormalization of the basis of the grading subspaces.

We say that the contraction is trivial if either J^ε is isomorphic to J, in cases where $\varepsilon_{ij} = 1$ or J^ε is a Jordan algebra with trivial multiplication, i.e. every $\varepsilon_{ij} = 0$. We then write $\varepsilon = (0)$. It is easy to check that both these cases satisfy (18). Obviously in both cases J^ε is a Jordan algebra.

In general we shall not be interested in the details of the structure of J, but we will need to specify whether $J_i \cdot J_j$ is identically zero or not. As in the case of Lie algebras, see [1], we introduce the symmetric matrix $\kappa = (\kappa_{ij})$:

$$\kappa_{ij} = \begin{cases} 0 & \text{if} \quad J_i \cdot J_j = 0 \\ 1 & \text{if} \quad J_i \cdot J_j \neq 0. \end{cases}$$

Consequently we can write

$$J_i \cdot J_j \subseteq \kappa_{ij} J_{i+j}$$

and speak of a Jordan algebra J with G-structure κ. The contraction is then determined by the matrices κ and ε. In general, the problem of determining the non-trivial contractions ε has to be solved for each G-graded structure κ. Note that we have chosen to consider the most general case of $\kappa = (1)$, the *generic case* where no $J_i \cdot J_j$ vanishes identically in (3).

To provide J^ε with the structure of a Jordan algebra the matrix ε of the contraction parameters (contraction matrix or just contraction for short) must violate neither the Jordan identity (2) nor commutativity (1).

In the case of generic contraction, J^ε is again a Jordan algebra if (1) and (2) are satisfied:

$$\begin{aligned}
&\varepsilon_{ij} J_i \cdot J_j = \varepsilon_{ji} J_j \cdot J_i \\
&\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{k+l,j+i}(J_i \cdot J_j)\cdot(J_k \cdot J_l) - \varepsilon_{kl}\varepsilon_{j,k+l}\varepsilon_{i,j+k+l} J_i \cdot (J_j \cdot (J_k \cdot J_l)) \\
&\qquad = \varepsilon_{ki}\varepsilon_{lj}\varepsilon_{l+j,k+i}(J_k \cdot J_i)\cdot(J_l \cdot J_j) - \varepsilon_{ki}\varepsilon_{k+i,j}\varepsilon_{k+i+j,l} J_l \cdot (J_j \cdot (J_k \cdot J_i)) \\
&\qquad = \varepsilon_{il}\varepsilon_{kj}\varepsilon_{i+l,k+j}(J_i \cdot J_j)\cdot(J_k \cdot J_l) - \varepsilon_{il}\varepsilon_{i+l,j}\varepsilon_{i+l+j,k} J_k \cdot (J_j \cdot (J_l \cdot J_i))
\end{aligned} \tag{19}$$

as a shorthand we wrote these expressions in the form of a graded summand of J.

The equalities can hold simultaneously for any choice of elements of the corresponding subspaces only if one has

$$\varepsilon_{ij} = \varepsilon_{ji} \qquad \varepsilon_{ij}\varepsilon_{kl}\varepsilon_{k+l,j+i} = \varepsilon_{kl}\varepsilon_{j,k+l}\varepsilon_{i,j+k+l} = \varepsilon_{ki}\varepsilon_{lj}\varepsilon_{l+j,k+i} \tag{20}$$

for all $k, i, j, l \in G$.

Let us make a final remark about non-generic cases. Suppose we have the G-grading of Jordan which is non-generic. For all i, j such that $J_i \cdot J_j = 0$ contraction parameter ε_{ij} is not well defined, so in order to obtain a system of equations corresponding to this G-structure κ one needs to exclude from (19) all terms with ε_{ij}.

Any contraction is given by a map $\varepsilon : G \times G \to \mathbb{C}$, with $(i, j) \to \varepsilon_{i,j}$ subject to equation (20). The set of such maps is closed under pointwise multiplication. Let us recall some facts from [2] about contractions of a Lie algebra over $\mathbb{C}$. Suppose that L is a G-graded Lie algebra over $\mathbb{C}$, then any contraction of Lie algebra L is given by $\varepsilon : G \times G \to \mathbb{C}$ such that for any $i, j, k \in G$

$$\varepsilon_{ij}\varepsilon_{i+j,k} = \varepsilon_{j,k}\varepsilon_{i,j+k}. \tag{21}$$

Moreover it was proved that contractions form a semigroup of 2-weak cohomology classes $H^2(G, \mathbb{C})$ on G with coefficients in the field $\mathbb{C}$. One may check that any solution of (21) also satisfies (20). It follows immediately that any map $\varepsilon : G \times G \to \mathbb{C}$ which gives a contraction of G-graded Lie algebra will also define a contraction for G-graded Jordan algebra or that any element of $H^2(G, \mathbb{C})$ gives a contraction of G-graded Jordan algebra.

One observes that, instead of (2) and (1), the analogous super-Jordan identities (5) and super-commutativity (4), would result in the same equation. Non-trivial solutions of (20) determine the contractions in the generic case.

5. Examples of contractions of Jordan algebras and superalgebras

In this section we consider contractions of simple Jordan algebras and for the simplest cases of $G : \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2$. We would like to remark here that algebras which we get after contractions of simple Jordan algebras could be interesting not only as illustrations of the method of graded contractions but also as a source of Jordan non-simple algebras.

5.1. Contraction in the case $G = \mathbb{Z}_2$

We consider any Jordan algebra J of finite or infinite dimension which is graded by the cyclic group $\mathbb{Z}_2$ of two elements. Thus we have

$$J = J_0 \oplus J_1 \qquad 0 \neq J_i \cdot J_j \subseteq J_{i+j} \qquad j, i, i + j \,(\mathrm{mod}\ 2).$$

The most general contraction $J \to J^\varepsilon$ of J that preserves the $\mathbb{Z}_2$-grading is described in terms of the matrix $\varepsilon = (\varepsilon_{ij}) \in \mathbb{C}^{2\times 2}$ of contraction parameters

$$J_i \cdot^\varepsilon J_j := \varepsilon_{ij} J_i \cdot J_j \subseteq \varepsilon_{ij} J_{i+j} \qquad i, j = 0, 1. \tag{22}$$

Here the subscript ε denotes the contracted multiplication. We will introduce the following convention: whenever $\varepsilon_{00} \neq 0$ we renormalize the basis of the grading subspace J_0 such that $\varepsilon_{00} = 1$.

The Jordan algebra J and its contraction J^ε are isomorphic as linear spaces, only the multiplication relations in J^ε are modified by (22). Equations (20) for J^ε being Jordan in the $\mathbb{Z}_2$ case give

$$\begin{aligned} &\varepsilon_{01} = \varepsilon_{10} && \varepsilon_{10}\left(\varepsilon_{00}^2 - \varepsilon_{10}\varepsilon_{i0}\right) = 0 \\ &\varepsilon_{11}^2\varepsilon_{00} = \varepsilon_{11}^2\varepsilon_{10} && \varepsilon_{11}\left(\varepsilon_{00}^2 - \varepsilon_{10}\varepsilon_{i0}\right) = 0 \end{aligned} \tag{23}$$

where $i = 0, 1$. They are solved trivially either by

$$\varepsilon = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{or by} \qquad \varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the cases of non-contraction or trivial contraction. The non-trivial contractions are given by the remaining solutions. There are three non-equivalent non-trivial cases:

$$\varepsilon_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \varepsilon_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Comparing these results with the case of $\mathbb{Z}_2$-graded contractions of Lie algebra [1] one may remark that we get the same non-equivalent contraction as in the Lie case. We illustrate this result by calculating $\mathbb{Z}_2$-contractions for concrete Jordan algebras.

Example 5.1. Let $J = \mathrm{Symm}(3, \mathbb{C})$ be a Jordan algebra of symmetric matrices 3×3 over $\mathbb{C}$. The $\mathbb{Z}_2$-grading (14) of $M_3(\mathbb{C})^{(+)}$ induces $\mathbb{Z}_2$-grading of $J = J_0 \oplus J_1$, where

$$J_0 = \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ c & 0 & d \end{pmatrix} \qquad J_1 = \begin{pmatrix} 0 & g & 0 \\ g & 0 & f \\ 0 & f & 0 \end{pmatrix}$$

with $a, b, c, d, f, g \in \mathbb{C}$.

Contracting the algebra $\mathrm{Symm}(3, \mathbb{C})$ by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ we obtain the following algebras:

- $J^{\varepsilon_1} = \left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right) \oplus \mathbb{C}e \oplus V$, where $e^2 = e$ and $V = \langle v_0, v_1 \rangle$ is a two-dimensional module over semisimple part $\mathrm{Symm}(2, \mathbb{C}) \oplus \mathbb{C}\mathrm{e}$ such that $E_{11}v_1 = ev_1 = \frac{1}{2}v_1, E_{22}v_0 = ev_0 = \frac{1}{2}v_0$ and $(E_{12} + E_{21})v_i = \frac{1}{2}v_{i+1}, i (\mathrm{mod}\ 2)$. By E_{ij} we denote a matrix in which the only non-zero entry is 1 on the ith row and the jth column.
- $J^{\varepsilon_2} = \left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right) \oplus \mathbb{C}e \oplus V$, where $e^2 = e$ and V is a two-dimensional module annihilated by the semi-simple part $\mathrm{Symm}(2, \mathbb{C}) \oplus \mathbb{C}\mathrm{e}$.
- $J^{\varepsilon_3} = A^{(+)} \oplus V$, where $A^{(+)} = \langle x, y \rangle / \langle x, y \rangle^3$ is a three generated nilpotent associative commutative algebra and V is a one-dimensional bimodule annihilated by $A^{(+)}$.

Example 5.2. Let $M_3(\mathbb{O})$ be the algebra of all 3×3 matrices with elements in octonions $\mathbb{O}$ with standard involution $x \to \bar{x}$. Consider the 27-dimensional simple Jordan algebra $J = \mathrm{Herm}(3, \mathbb{O})$ over $\mathbb{C}$. Any element of J may be written as

$$X = \begin{pmatrix} \alpha & a & \bar{b} \\ \bar{a} & \beta & c \\ b & \bar{c} & \gamma \end{pmatrix}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $a, b, c \in \mathbb{O}$. The grading $M_3(\mathbb{O})^{(+)}$ induces the $\mathbb{Z}_2$-grading of $\mathrm{Herm}(3, \mathbb{O})$,

$$J_0 = \begin{pmatrix} \alpha & 0 & \bar{b} \\ 0 & \beta & 0 \\ b & 0 & \gamma \end{pmatrix} \qquad J_1 = \begin{pmatrix} 0 & a & 0 \\ \bar{a} & 0 & c \\ 0 & \bar{c} & 0 \end{pmatrix}.$$

Under contraction we get:

- $J^{\varepsilon_1} = \left(\begin{smallmatrix} \alpha & \bar{b} \\ b & \gamma \end{smallmatrix}\right) \oplus \mathbb{C}e \oplus V$, where $e^2 = e$ and $V = \mathbb{O}v_1 \oplus \mathbb{O}v_2$ is a 16-dimensional module over a semisimple part with $E_{11}v_1 = ev_1 = \frac{1}{2}v_1, E_{22}v_0 = ev_0 = \frac{1}{2}v_0$ and $(bE_{21} + \bar{b}E_{12})v_i = b\bar{v}_{i+1}, i (\mathrm{mod}\ 2)$.
- $J^{\varepsilon_2} = \left(\begin{smallmatrix} \alpha & \bar{b} \\ b & \gamma \end{smallmatrix}\right) \oplus \mathbb{C}e \oplus V$, where $e^2 = e$ and V is a 16-dimensional module annihilated by the semisimple part $\mathrm{Herm}(2, \mathbb{O}) \oplus \mathbb{C}e$ (which is a subalgebra in J).
- $J^{\varepsilon_3} = J' \oplus V$, where $J' = \mathbb{O}[x, y]/\langle x^2 - 1, y^2 - 1, x^3, y^3 \rangle$ is a commutative nilpotent 26-dimensional algebra and V is a one-dimensional bimodule annihilated by J'.

In particular, one notes that all contracted Jordan algebras are special.

Example 5.3. Let $J_t = J_0 \oplus J_1 = \langle 1, a\rangle \oplus \langle \xi, \eta\rangle$ be a simple Jordan superalgebra such that $a \cdot \xi = \xi, a \cdot \eta = \eta, a^2 = 2a, \xi^2 = \eta^2 = 0$ and $\xi \cdot \eta = 1 + ta$, where $t \in \mathbb{C}, t \neq -\frac{1}{2}$. Then we get the following contractions of J_t:

- $J_t^{\varepsilon_1} \simeq \mathbb{C} \oplus \mathbb{C} \oplus \langle v_1, v_2\rangle$, is a Jordan superalgebra with $J_0 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $e_i v_j = 0$, $v_j^2 = 0$ for all $i, j \in \{0, 1\}$.
- $J_t^{\varepsilon_2} \simeq J_0' \oplus J_1$, where $J_0' = \langle \xi \cdot \eta, v\rangle$ such that $\xi v = \eta v = 0$.
- $J_t^{\varepsilon_3} \simeq J_0 \oplus \langle \xi, \eta\rangle$, where $\langle \xi, \eta\rangle$ is a Jordan algebra with a trivial multiplication.

5.2. Contraction in the case $G = \mathbb{Z}_3$

Any $\mathbb{Z}_3$ Jordan algebra may be decomposed into

$$J = J_0 \oplus J_1 \oplus J_2 \qquad 0 \neq J_i \cdot J_j \subseteq J_{i+j} \qquad j, i, i + j \,(\mathrm{mod}\ 3).$$

Any generic contraction is described in terms of matrix $\varepsilon = (\varepsilon_{ij}) \in \mathbb{C}^{3\times 3}$. In order for the algebra J^ε to remain Jordan we deduce from equation (20) equations for the case of $\mathbb{Z}_3$-grading:

$$\begin{array}{ll} \varepsilon_{ij} = \varepsilon_{ji} & \varepsilon_{i0}^3 = \varepsilon_{i0}^2\varepsilon_{00} = \varepsilon_{i0}\varepsilon_{00}^2 \\ \varepsilon_{l1}(\varepsilon_{00}\varepsilon_{i0} - \varepsilon_{j0}\varepsilon_{k0}) = 0 & \varepsilon_{11}\varepsilon_{12}\varepsilon_{00} = \varepsilon_{11}\varepsilon_{12}\varepsilon_{i0} = \varepsilon_{11}^2\varepsilon_{22} \\ \varepsilon_{22}\varepsilon_{12}\varepsilon_{00} = \varepsilon_{22}\varepsilon_{12}\varepsilon_{i0} = \varepsilon_{22}^2\varepsilon_{11} & \varepsilon_{21}^3 = \varepsilon_{21}^2\varepsilon_{i0} \end{array} \tag{24}$$

where $i, j, k, l = 1, 2$. The solutions of these equations give 13 non-trivial non-equivalent contractions:

$$\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\varepsilon_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \varepsilon_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varepsilon_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varepsilon_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \varepsilon_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \varepsilon_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_{13} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let us make two remarks here. First, if we check the results from [1] we get the same picture for $\mathbb{Z}_3$-contractions as for the case of Jordan algebras as for $\mathbb{Z}_3$-contraction for Lie algebras. Second, one observes that contractions therein derived are symmetric with respect to J_1 and J_2. Therefore we will consider only one of each pair of cases above which differ by interchanging $J_1 \leftrightarrow J_2$.

We calculate some contractions of $\mathbb{Z}_3$-graded Jordan algebra $M_3(\mathbb{C})^{(+)}$.

Example 5.4. $J = M_3(\mathbb{C})^{(+)}$. In section 3 (16) we get a $\mathbb{Z}_3$-grading for $J = J_0 \oplus J_1 \oplus J_2$. For this grading the contracted algebras will be the following:

- $J^{\varepsilon_1} = \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus V$, where V is a six-dimensional bimodule annihilated by semisimple part.
- $J^{\varepsilon_2} \simeq J^{\varepsilon_3} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus (V_{12} \oplus V_{13} \oplus V_{23}) \oplus V$, where V_{ij} is a one-dimensional module, such that $e_i v_{ij} = e_j v_{ij} = \frac{1}{2}v_{ij}$, $v_{ij} \in V_{ij}$, $i = 1, 2$, $j = 2, 3$ and V is three-dimensional bimodule annihilated by semisimple part.
- $J^{\varepsilon_4} \simeq J^{\varepsilon_8} = A^{(+)} \oplus V$, where $A^{(+)} = \langle x, y, z\rangle/\langle x^2, y^2, z^2, xyz\rangle$ is a three generated nilpotent associative commutative algebra and V is three-dimensional bimodule annihilated by $A^{(+)}$.
- $J^{\varepsilon_5} \simeq J^{\varepsilon_9} = A^{(+)} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, where $A^{(+)} = \langle x, y, z\rangle/\langle x^2, y^2, z^2, xyz\rangle$ is a nilpotent associative commutative algebra with three generators.
- $J^{\varepsilon_6} = A^{(+)} \oplus A^{(+)} \oplus A^{(+)}$, where $A^{(+)} = \langle x, y\rangle/\langle x^2, y^2\rangle$ and V is a three-dimensional bimodule annihilated by $A^{(+)}$.
- $J^{\varepsilon_7} \simeq J^{\varepsilon_{10}} = \mathbb{C}\langle x, y, z\rangle$, non-associative Grassman algebra.
- $J^{\varepsilon_{11}} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus (V_{12} \oplus V_{13} \oplus V_{23})^2$, where V_{ij} and e_i are as in J^{ε_2} and V.

Example 5.5. $J = J(V, f_1)$ (example from section 3). We demonstrated there that J admits the following $\mathbb{Z}_3$-grading $J = \langle 1, v_3\rangle \oplus \langle v_1\rangle \oplus \langle v_2\rangle$. This grading is a non-generic grading $\left(J_1^2 = J_2^2 = 0\right)$. In the case of non-generic gradings some of the terms in (19) would not be present. Therefore the $\mathbb{Z}_3$-graded contractions of Jordan algebras with non-generic grading structures are determined by a subset of equations (24). The solutions of (24) will also be the solutions for the subsystem defined by the non-generic grading. We write some contractions for J

- $J^{\varepsilon_1} \simeq J^{\varepsilon_5} \simeq J^{\varepsilon_9} \simeq \mathbb{C} \oplus \mathbb{C} \oplus V$, where V is a two-dimensional bimodule annihilated by semi-simple part.
- $J^{\varepsilon_4} \simeq J^{\varepsilon_8} = (J, 0)$, an algebra with zero multiplication.
- $J^{\varepsilon_6} \simeq J^{\varepsilon_7} \simeq J^{\varepsilon_{10}} \simeq S \oplus V$, where $S = \langle x, y\rangle/\langle x^2, y^2\rangle$ is an associative nilpotent algebra and V is a one-dimensional module annihilated by S.
- $J^{\varepsilon_{11}} \simeq J^{\varepsilon_{12}} \simeq J^{\varepsilon_{13}} \simeq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \langle v_1, v_2\rangle$, where e_1, e_2 are idempotents and $e_j v_i = \frac{1}{2}v_i$, for all $i, j = 1, 2$.

5.3. Contraction in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

The graded group in this section is the tensor product of two cyclic groups of order 2. We consider any Jordan algebra or superalgebra which admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading. We then may decompose J into

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11} \qquad \text{such that} \quad J_{ij} \cdot J_{kl} \subset J_{i+k,j+l}$$

where the subscripts have two components, each read modulo 2. Again we restrict ourselves to considering the generic case. Then matrix ε can be written as

$$\varepsilon = \begin{pmatrix} \varepsilon_{00,00} & \varepsilon_{00,01} & \varepsilon_{00,10} & \varepsilon_{00,11} \\ \varepsilon_{01,00} & \varepsilon_{01,01} & \varepsilon_{01,10} & \varepsilon_{01,11} \\ \varepsilon_{10,00} & \varepsilon_{10,01} & \varepsilon_{10,10} & \varepsilon_{10,11} \\ \varepsilon_{11,00} & \varepsilon_{11,01} & \varepsilon_{11,10} & \varepsilon_{11,11} \end{pmatrix}.$$

In this case we will not specify the equations of (20) for $\mathbb{Z}_2 \times \mathbb{Z}_2$. The reason why we do not write the equations is that, first, $\mathbb{Z}_2\mathbb{Z}_2$ is rich enough to provide a large system of equations and, second, there are 41 non-equivalent contractions, including two trivial ones and they are exactly the same as non-trivial contractions for the case of Lie algebra $\mathbb{Z}_2 \times \mathbb{Z}_2$-contractions. All of them are given in two tables in [1], therefore we refrain from writing them here. Rather than trying to solve (20) for $\mathbb{Z}_2 \times \mathbb{Z}_2$ directly, one may first observe that this system is solved by tensor products of $\mathbb{Z}_2$-solutions (for the generic case). As an example, starting from three non-trivial $\mathbb{Z}_2$-solutions $\varepsilon_1, \varepsilon_2, \varepsilon_3$ we got

$$\varepsilon_1 \otimes \varepsilon_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_1 \otimes \varepsilon_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \varepsilon_2 \otimes \varepsilon_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

Example 5.6. $J = \mathrm{Symm}(3, \mathbb{C})$. From section 3 we have $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading

$$J = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \oplus \begin{pmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & f & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ g & 0 & 0 \end{pmatrix}$$

For the contractions calculated above we will have:

- $J^{\varepsilon_1 \otimes \varepsilon_2} \simeq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \langle v_1 \rangle \oplus V$, where e_i are idempotents, V is a two-dimensional bimodule annihilated by semi-simple part, $e_1 v_1 = e_2 v_1 = \frac{1}{2} v_1$.
- $J^{\varepsilon_1 \otimes \varepsilon_3} \simeq \langle x, y \rangle / \langle x^2, y^3 \rangle \oplus V_2$, where $\langle x, y \rangle$ is an associative commutative algebra and V is a two-dimensional bimodule annihilated by $\langle x, y \rangle / \langle x^2, y^3 \rangle$.
- $J^{\varepsilon_2 \otimes \varepsilon_3} \simeq \langle x, y \rangle / \langle x^3 \rangle \oplus V$, where V is a four-dimensional bi-module annihilated by $\langle x, y \rangle / \langle x^3 \rangle$.

6. Contractions of representations of Jordan algebras

In this section we simultaneously define contractions for bimodules as well as for special modules of Jordan algebra.

6.1. Contractions of Jordan modules

Suppose that J is a G-graded Jordan algebra. We set G-Mod(J) to be a category of G-graded bimodules of J, i.e. J-bimodule V such that

- V is graded by G
- $J_i \cdot V_j \subseteq V_{i+j} \qquad i, j \in G.$

Let ε be a contraction matrix for J, what means a map satisfying (20). Now we want V to be a J^ε-bimodule. Let us consider a set of all maps $\phi : G \times G \to \mathbb{C}$, such that

$$\begin{aligned} \varepsilon_{ij}\phi_{km}\phi_{i+j,m+k} &= \varepsilon_{ij}\phi_{i+j,m}\phi_{k,i+j+m} = \varepsilon_{jk}\phi_{im}\phi_{j+k,m+i} \\ \varepsilon_{ik}\varepsilon_{i+k,j}\phi_{i+j+k,m} &= \varepsilon_{ik}\phi_{jm}\phi_{i+k,j+m} = \phi_{im}\phi_{j,m+i}\phi_{k,m+i+j} \\ &= \varepsilon_{jk}\phi_{im}\phi_{j+k,m+i} = \phi_{km}\phi_{j,m+k}\phi_{i,m+k+j} \end{aligned} \tag{26}$$

for all $i, j, k, m \in G$.

Consequently, given ϕ satisfying (26) and $V \in G$-Mod(J) we define V^ϕ to be the J^ε-modules with

- vector space structure equal to V;
- action of J^ε defined by

$$\begin{aligned} x \cdot^\phi v &= \phi_{i,m} x \cdot v \qquad & x \in (J^\varepsilon)_i, v \in V_m \\ v \cdot^\phi x &= \phi_{m,i} v \cdot x \qquad & x \in (J^\varepsilon)_i, v \in V_m. \end{aligned}$$

Both equations (26) provide that a new action J^ε on V satisfies (6). Thus V^ϕ is a J^ε-bimodule.

6.2. Contraction of Jordan special modules

Let G-SMod(J) be a category of special G-graded modules of a Jordan algebra J. We consider the set of all maps $\phi : G \times G \to \mathbb{C}$ such that

$$\varepsilon_{ij}\phi_{i+j,m} = \phi_{j,i+m}\phi_{im} = \phi_{jm}\phi_{i,j+m} \tag{27}$$

for all $i, j, k, m \in G$.

Again, given ϕ satisfying (27) and a module $W \in G$-SMod(J). we define on vector space W a structure of special J^ε-module W^ϕ by action

$$w \cdot^\phi x = \phi_{i,m} w \cdot x \qquad x \in (J^\varepsilon)_i, w \in W_m.$$

Equation (27) provides W^ϕ becomes a special Jordan J^ε-module. Moreover, we remark that solutions of the equations (27) for Jordan spatial modules coincide with the solutions of the equations for contractions of Lie modules for Lie algebras [2]. Therefore, if we have a contraction matrix of a G-graded Lie algebra (which will automatically be a contraction matrix for a G-graded Jordan algebra), then the allowed contraction matrices for special modules for the Jordan algebra case coincide with the contractions for the Lie-module case. In particular, from [2] tables 1 and 2 we get the whole picture for the case $\mathbb{Z}_2$-, $\mathbb{Z}_3$-gradings respectively.

7. Examples of contractions of *G*-graded representations of contracted Jordan algebras

In this section, we consider the contractions of representations for contracted algebras mostly from section 5.

7.1. Contractions in the case $G = \mathbb{Z}_2$

Suppose $V \in \mathbb{Z}_2$-Mod(J) then the action of a $\mathbb{Z}_2$-graded J splits V into the direct sum

$$V = V_0 \oplus V_1$$

where the subspaces V_i, $i = 0, 1$ are defined by the grading properties:

$$0 \neq J_i V_m \subseteq V_{i+m} \qquad \text{and} \qquad 0 \neq V_m J_i \subseteq V_{i+m} \qquad i, m, i+m (\mathrm{mod}\, 2).$$

As above we assume the generic situation. We may also write V and JV as follows:

$$V = \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \quad \text{and} \quad JV = \begin{pmatrix} J_0 & J_1 \\ J_1 & J_0 \end{pmatrix} \cdot \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0 V_0 + J_1 V_1 \\ J_1 V_0 + J_0 V_1 \end{pmatrix}. \tag{28}$$

After contracting J by ε and V by ϕ we have

$$J^\varepsilon \cdot^\phi V = \begin{pmatrix} J_0 & J_1 \\ J_1 & J_0 \end{pmatrix}^\varepsilon \cdot^\phi \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} \phi_{00} J_0 & \phi_{11} J_1 \\ \phi_{10} J_1 & \phi_{01} J_0 \end{pmatrix} \cdot \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} \phi_{00} J_0 V_0 + \phi_{11} J_1 V_1 \\ \phi_{01} J_0 V_1 + \phi_{10} J_1 V_0 \end{pmatrix}.$$

In particular, for the adjoint action of J^ε on itself, we have $\phi = \varepsilon$. Since we want V^ϕ to be a Jordan bimodule for J^ε, ϕ has to satisfy (26) for the $\mathbb{Z}_2$-case:

$$\begin{array}{lll} \phi_{0i}^3 = \phi_{0i}^2 \varepsilon_{00} = \phi_{0i} \varepsilon_{00}^2 & \varepsilon_{11}\phi_{01}\varepsilon_{10} = \varepsilon_{11}\phi_{10}\phi_{0i} = \phi_{10}^2 \phi_{11} & \\ \varepsilon_{11}\phi_{11}(\phi_{00} - \phi_{01}) = 0 & \phi_{10}(\gamma_{0i}\phi_{0j} - \varepsilon_{01}\varepsilon_{k0}) = 0 & \phi_{11}(\gamma_{0i}\phi_{0j} - \varepsilon_{01}\varepsilon_{k0}) = 0 \end{array} \tag{29}$$

for $i, j, k = 0, 1$ and $\gamma \in \{\phi, \varepsilon\}$

Similarly, if we want $V^\phi \in \mathbb{Z}_2$-SMod(J) then ϕ satisfies (27) for the Z_2-case.

$$\begin{array}{lll} \varepsilon_{00}\phi_{0i} = \phi_{0i}^2 & \phi_{10}\phi_{0i} = \varepsilon_{10}\phi_{10} & \\ \phi_{11}\phi_{0i} = \phi_{11}\varepsilon_{10} & \varepsilon_{11}\phi_{0i} = \phi_{11}\phi_{10} & i = 0, 1. \end{array} \tag{30}$$

Table 1. Non-trivial $\mathbb{Z}_2$-graded contractions of representations in the form (28).

ε	No	$J \cdot^{\varphi} V$	
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	I.1	$\begin{pmatrix} J_0 & 0 \\ J_1 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0V_0 \\ J_1V_0 + J_0V_1 \end{pmatrix}$	$\varepsilon = \varphi$
	I.2	$\begin{pmatrix} J_0 & J_1 \\ 0 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0V_0 + J_1V_1 \\ J_0V_1 \end{pmatrix}$	
	I.3	$\begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0V_0 \\ J_0V_1 \end{pmatrix}$	
	I.4	$\begin{pmatrix} J_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0V_0 \\ 0 \end{pmatrix}$	
	I.5	$\begin{pmatrix} 0 & 0 \\ 0 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} 0 \\ J_0V_1 \end{pmatrix}$	
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	II.1	$\begin{pmatrix} 0 & J_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_1V_1 \\ 0 \end{pmatrix}$	$\varepsilon = \varphi$
	II.2	$\begin{pmatrix} 0 & 0 \\ J_1 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} 0 \\ J_1V_0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	III.1	$\begin{pmatrix} J_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0V_0 \\ 0 \end{pmatrix}$	$\varepsilon = \varphi$
	III.2	$\begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_0V_0 \\ J_0V_1 \end{pmatrix}$	
	III.3	$\begin{pmatrix} 0 & 0 \\ 0 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} 0 \\ J_0V_1 \end{pmatrix}$	
	III.4	$\begin{pmatrix} 0 & J_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} J_1V_1 \\ 0 \end{pmatrix}$	
	III.5	$\begin{pmatrix} 0 & 0 \\ J_1 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = (J_1V_0)$	

Solutions of equations (29) and (30) give sets of all $\mathbb{Z}_2$-contraction matrices for Jordan modules and Jordan special modules respectively. It is clear that ϕ strictly depends on ε. It turns out that for any of the three non-trivial contractions ε of Jordan algebras, solutions of (26) and of (27) coincide, moreover they also coincide with the solutions for the analogous equations for the $\mathbb{Z}_2$-graded representations of Lie algebra, see [1]. Table 1 gives a complete list of solutions in the form (28).

Example 7.1. Consider an example of the simple Jordan algebra $J = \mathrm{Symm}(2, \mathbb{C})$, the Jordan algebra of 2×2 complex symmetric matrices, and its special Jordan module $V = M_2(\mathbb{C})^{(+)}$, the space of 2×2 complex matrices. The standard basis of the algebra is given by the matrices $E_{11}, E_{22}, h := E_{12} + E_{21}$, with E_{11}, E_{22} being orthogonal idempotents and

$$E_{11} \cdot h = E_{22} \cdot h = \tfrac{1}{2}h \qquad h^2 = E_{11} + E_{22}.$$

Consider the following $\mathbb{Z}_2$-grading of J:

$$J_0 = \mathbb{C}E_{11} + \mathbb{C}E_{22} \qquad J_1 = \mathbb{C}h$$

where $\mathbb{C}$ stands for an arbitrary complex coefficient. Then one can verify that

$$J_0 \cdot J_0 = J_0 \qquad J_0 \cdot J_1 = J_1 \qquad J_1 \cdot J_1 = \mathbb{C}h \subset J_0.$$

Consequently the grading is generic. The representation space V may also be graded

$$V_0 = \mathbb{C}F_{11} + \mathbb{C}F_{22} V_1 = \mathbb{C}F_{21} + \mathbb{C}F_{12}$$

where F_{ij} denote a standard basis of $V = M_2(\mathbb{C})$.

In this very simple example it is equally easy to write all relations in matrix form,

$$J_0V_0 = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} ak_1 & 0 \\ 0 & ck_2 \end{pmatrix} \quad J_0V_1 = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & an_2 \\ cn_1 & 0 \end{pmatrix}$$

$$J_1V_0 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} 0 & bk_2 \\ bk_1 & 0 \end{pmatrix} \quad J_1V_1 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix} = \begin{pmatrix} bn_1 & 0 \\ 0 & bn_2 \end{pmatrix}$$

for all $a, b, c, k_1, k_2, n_1, n_2$.

We consider the contraction of J by ε_1. The entries I.1–I.5 of table 1 are the contractions of the representations in this case. Thus, for I.1 we get

$$J \cdot^{\phi} V = \begin{pmatrix} J_0 & 0 \\ J_1 & J_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & c \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \\ 0 & n_2 \\ n_1 & 0 \end{pmatrix} = \begin{pmatrix} ak_1 & 0 \\ 0 & ck_2 \\ 0 & bk_2 + an_2 \\ bk_1 + cn_1 & 0 \end{pmatrix}.$$

Without loss of generality one may rewrite it as

$$J \cdot^{\phi} V = \begin{pmatrix} ak_1 & bk_2 + an_2 \\ bk_1 + an_1 & ck_2 \end{pmatrix}.$$

Similarly for I.2 of the table one finds

$$J \cdot^{\phi} V = \begin{pmatrix} J_0 & J_1 \\ 0 & J_0 \end{pmatrix} = \begin{pmatrix} ak_1 + bn_1 & an_2 \\ cn_1 & ck_2 + bn_2 \end{pmatrix}.$$

Example 7.2. Let J be a Jordan algebra. Consider the contraction of J given by $\varepsilon_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In this case (29) becomes

$$\phi_{00} = \phi_{01} = \phi_{10}\phi_{11} = 0. \tag{31}$$

One may easily check that for the case of a special Jordan module the system (30) becomes (31). There are two possibilities for ϕ in this particular case:

$$\phi_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \qquad \text{and} \qquad \phi_2 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

In the form of (28) we write the contracted linear transformations as

$$\begin{pmatrix} 0 & aJ_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} aJ_1V_1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ bJ_1 & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} 0 \\ bJ_1V_0 \end{pmatrix}. \tag{32}$$

One can verify directly that (32) defines, indeed, representations of contracted Jordan algebra J^{ε}.

7.2. Contractions for the case $G = \mathbb{Z}_3$

Suppose $V \in \mathbb{Z}_3$-Mod(J) then the action of a $\mathbb{Z}_3$-graded J splits V into the direct sum

$$V = V_0 \oplus V_1 \oplus V_2$$

where the subspaces V_i, $i = 0, 1, 2$ are defined by the grading properties:

$$0 \neq J_iV_m \subseteq V_{i+m} \qquad \text{and} \quad 0 \neq V_mJ_i \subseteq V_{i+m} \qquad i, m, i+m (\text{mod } 3).$$

We may write it in the form of (28)

$$J^{\varepsilon} \cdot^{\phi} V = \begin{pmatrix} \phi_{00}J_0V_0 + \phi_{21}J_2V_1 + \phi_{12}J_1V_2 \\ \phi_{10}J_1V_0 + \phi_{01}J_0V_1 + \phi_{22}J_2V_2 \\ \phi_{20}J_2V_0 + \phi_{11}J_1V_1 + \phi_{02}J_0V_2 \end{pmatrix}.$$

If we want V^{ϕ} to be a Jordan bimodule for J^{ε} then we need ϕ satisfying (26) for $\mathbb{Z}_3$:

$$
\begin{aligned}
&\varepsilon_{00}^2\phi_{00} = \varepsilon_{00}\phi_{00}^2 = \phi_{00}^3 \\
&\varepsilon_{00}^2\phi_{0i} = \varepsilon_{00}\phi_{0i}^2 = \phi_{0i}^3 \\
&\phi_{i0}\gamma_{00}\phi_{00} = \phi_{i0}\gamma_{0i}\gamma_{0i} = \phi_{0i}\gamma_{00}\gamma_{0i} \\
&\varepsilon_{ii}\phi_{\bar{i}0}\gamma_{0i} = \varepsilon_{ii}\phi_{\bar{i}0}\phi_{00} = \gamma_{0i}\phi_{ii}\phi_{i0} = \phi_{ii}\phi_{i0}\phi_{00} = \varepsilon_{ii}\varepsilon_{\bar{i}0}\phi_{\bar{i}0} \\
&\phi_{ii}\gamma_{0i}\gamma_{0i} = \phi_{ii}\gamma_{0i}\varepsilon_{00} = \phi_{ii}\phi_{0\bar{i}}\varepsilon_{00} = \phi_{ii}\phi_{0\bar{i}}\gamma_{0i} = \phi_{ii}\phi_{0\bar{i}}^2 \\
&\phi_{12}\phi_{00}\gamma_{00} = \phi_{12}\phi_{0\bar{i}}\gamma_{00} = \varepsilon_{0i}\phi_{12}\gamma_{00} = \phi_{12}\varepsilon_{0i}^2 = \phi_{12}\phi_{0\bar{i}}^2 \\
&\phi_{ij}\varepsilon_{11}\varepsilon_{22} = \phi_{ij}\varepsilon_{12}\varepsilon_{i0} = \phi_{ij}\phi_{20}\phi_{12} = \phi_{ij}\phi_{10}\phi_{21} = \phi_{ij}\phi_{11}\phi_{22} \\
&\qquad\quad = \phi_{ij}\varepsilon_{12}\phi_{0,i+j} = \phi_{ij}\varepsilon_{12}\phi_{0j} = \varepsilon_{ii}\phi_{\bar{i}i}\phi_{\bar{i},i+\bar{j}} \\
&\varepsilon_{ii}\phi_{\bar{i}i}\gamma_{0i} = \phi_{\bar{i}i}\varepsilon_{ii}\phi_{00} = \phi_{\bar{i}i}\varepsilon_{ii}\varepsilon_{\bar{i}0} = \phi_{i\bar{i}}\phi_{ii}\phi_{0k} \qquad k = 0, 1 \\
&\varepsilon_{ii}\phi_{i\bar{i}}\varepsilon_{\bar{i}0} = \varepsilon_{ii}\phi_{i0}\phi_{\bar{i}\bar{i}} = \phi_{i0}\phi_{ii}\phi_{i\bar{i}} = \varepsilon_{ii}\varepsilon_{i\bar{i}}\phi_{0k} \qquad k = 0, 1, 2 \\
&\phi_{ii}\phi_{0k}\phi_{i0} = \phi_{i\bar{i}}\phi_{i0}\varepsilon_{i0} = \varepsilon_{ii}\phi_{\bar{i}\bar{i}}\gamma_{0k} = \phi_{i\bar{i}}\phi_{i0}\phi_{00} \qquad k = 1, 2 \\
&\varepsilon_{12}\phi_{k0}\phi_{0i} = \varepsilon_{ii}\phi_{\bar{i}0}\phi_{\bar{i}\bar{i}} = \phi_{0i}\phi_{\bar{i}i}\phi_{i0} = \phi_{i0}\phi_{ii}\phi_{\bar{i}\bar{i}} \\
&\qquad\quad = \phi_{i0}\phi_{\bar{i}0}\phi_{i\bar{i}} = \varepsilon_{ii}\varepsilon_{\bar{i}\bar{i}}\phi_{i0} = \varepsilon_{ii}\varepsilon_{0i}\phi_{i0} \qquad k = 0, i \\
&\varepsilon_{12}\phi_{0k}^2 = \varepsilon_{12}\varepsilon_{0k}\phi_{0m} = \varepsilon_{0l}\phi_{10}\phi_{21} = \varepsilon_{0l}\phi_{20}\phi_{12} = \gamma_{0l}\phi_{11}\phi_{22} = \phi_{12}\phi_{20}\phi_{02} \\
&\qquad = \phi_{12}\phi_{20}\phi_{00} = \phi_{21}\phi_{10}\phi_{01} = \phi_{21}\phi_{10}\phi_{00} \qquad k, m = 0, 1, 2 \quad l = 1, 2.
\end{aligned}
\tag{33}
$$

For all equations above $i = 1, 2$, $j = 1, 2$, $\bar{a} = -a$ and $\gamma \in \{\phi, \varepsilon\}$.

If we want V^{ϕ} to be a special Jordan module then from (27) we deduce the following system of equations for ϕ:

$$
\begin{array}{ll}
\varepsilon_{00}\phi_{0m} = \phi_{0m}^2 & \varepsilon_{01}\phi_{1m} = \phi_{1m}\phi_{0,m+1} = \phi_{1m}\phi_{0m} \\
\varepsilon_{02}\phi_{2m} = \phi_{2m}\phi_{0,m+2} = \phi_{2m}\phi_{0m} & \varepsilon_{11}\phi_{2m} = \phi_{1m}\phi_{1,m+1} \\
\varepsilon_{22}\phi_{1m} = \phi_{2m}\phi_{2,m+2} & \varepsilon_{12}\phi_{0m} = \phi_{2m}\phi_{1,m+2} = \phi_{1m}\phi_{2,m+1}
\end{array}
\tag{34}
$$

where $m = 0, 1, 2$ and subscripts are taken modulo 3.

As we have already mentioned, system (34) coincides with the system corresponding to ϕ for the case of representations of graded contractions for Lie algebras. Consequently, since we obtained the same set of non-equivalent contractions, we obtain the same solutions as in the case of Lie algebras. There is no need to rewrite the corresponding table, see table 2 in [2]. Further, the solutions of (34) are also among the solutions of (33). However, (33) admits other solutions. We will write them in table 2. There Jordan algebra contractions are given by ε. The contractions of representations are given by the corresponding ϕ. Out of three matrices ϕ, which differ by cyclic permutation of columns, only one is shown.

Example 7.3. Consider the contraction J^{ε} given by

$$
\varepsilon = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Let us consider two Jordan bimodule contractions

$$
\phi_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.
$$

Table 2. Non-trivial $\mathbb{Z}_3$-graded contractions of Jordan bimodules which are not contractions for special Jordan modules.

ε	No	φ	$J \cdot^{\varphi} V$
$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	I.1	$\begin{pmatrix}0&0&0\\1&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0+J_1V_1\end{pmatrix}$
	I.2	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\J_2V_0\end{pmatrix}$
	I.3	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_1V_1\end{pmatrix}$
	I.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&J_2&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_2V_1\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	II.1	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&J_2&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_2V_0\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}1&0&1\\0&0&0\\1&0&0\end{pmatrix}$	III.1	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_1V_1\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	IV.1	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_1V_1\end{pmatrix}$
	IV.2	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\0\end{pmatrix}$
	IV.3	$\begin{pmatrix}0&0&0\\-1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0\end{pmatrix}$
	IV.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	V.1	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}J_0&0&0\\0&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_0V_0\\J_2V_2\\0\end{pmatrix}$
	V.2	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_1V_1\end{pmatrix}$
	V.3	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0\end{pmatrix}$
	V.4	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\0\end{pmatrix}$
	V.5	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	VI.1	$\begin{pmatrix}0&0&0\\1&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0+J_1V_1\end{pmatrix}$
	VI.2	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\J_2V_0\end{pmatrix}$
	VI.3	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_1V_1\end{pmatrix}$
	VI.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&J_2&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_2V_1\\0\\J_2V_0\end{pmatrix}$

Table 2. (Continued.)

ε	No	φ	$J\cdot^{\varphi}V$
$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	VII.1	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_1V_1\end{pmatrix}$
	VII.2	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\0\end{pmatrix}$
	VII.3	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0\end{pmatrix}$
	VII.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$	VIII.1	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0\end{pmatrix}$
	VIII.2	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\0\end{pmatrix}$
	VIII.3	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\0\end{pmatrix}$
	VIII.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&J_2&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_2V_1\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$	IX.1	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}J_0&0&0\\0&0&0\\0&J_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_0V_0\\0\\J_1V_1\end{pmatrix}$
	IX.2	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\0\end{pmatrix}$
	IX.3	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0\end{pmatrix}$
	IX.4	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\0\end{pmatrix}$
	IX.5	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&J_2&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_2V_1\\0\\J_2V_0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&0&1\\0&1&1\end{pmatrix}$	X.1	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\0\end{pmatrix}$
	X.2	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&J_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0+J_2V_2\\0\end{pmatrix}$
	X.3	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\J_1&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}0\\J_1V_0\\J_2V_0\end{pmatrix}$
	X.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&J_2&0\\0&0&0\\J_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_2V_1\\0\\J_2V_0\end{pmatrix}$

Here ϕ_1 is an admissible contraction for both the Jordan and special Jordan modules, while ϕ_2 is a contraction matrix only in the case of a special module. In the form (28) we write the contracted linear transformations as

$$\begin{pmatrix}J_0&0&0\\J_1&J_0&0\\0&0&J_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}=\begin{pmatrix}J_0V_0\\J_1V_0+J_0V_1\\J_0V_2\end{pmatrix} \tag{35}$$

$$\begin{pmatrix} 0 & J_2 & 0 \\ 0 & 0 & 0 \\ J_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} J_2 V_1 \\ 0 \\ J_2 V_0 \end{pmatrix}. \tag{36}$$

One can verify directly that (35) is, indeed, a special representation of the contracted Jordan algebra J. We will check it for $J_0 \cdot_\varepsilon J_1 \subseteq J_1$

$$J_1 V = \begin{pmatrix} 0 & 0 & 0 \\ J_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ J_1 V_0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} J_1 V \supseteq (J_0 \cdot^\varepsilon J_1)^\phi V &= \tfrac{1}{2} J_0^\phi \big(J_1^\phi V\big) + \tfrac{1}{2} J_1^\phi \big(J_0^\phi V\big) \\ &= \frac{1}{2} \begin{pmatrix} J_0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & J_0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ J_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\ &\quad + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ J_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J_0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & J_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\ &= \frac{1}{2} \left(\begin{pmatrix} 0 & 0 & 0 \\ J_0 J_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ J_1 J_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ J_0 J_1 V_0 \\ 0 \end{pmatrix}. \end{aligned}$$

With (36) it is not the case, since

$$\begin{aligned} 0 = (J_2 \cdot^\varepsilon J_2)^\phi V &= J_2^\phi \big(J_2^\phi V\big) \\ &= \frac{1}{2} \begin{pmatrix} 0 & J_2 & 0 \\ 0 & 0 & 0 \\ J_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & J_2 & 0 \\ 0 & 0 & 0 \\ J_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & J_2 J_2 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \neq 0. \end{aligned}$$

Straightforward, but long calculations show that (36) will give a Jordan bimodule for J^ε.

8. Contractions of tensor product of representations

Let ε define a contraction for G-graded Jordan algebra and let V, W be G-graded Jordan special modules of J. Finally, let ϕ be a contraction of the G-graded Jordan special bimodule compatible with ε. Consider $V \otimes W$ graded by G in the usual way:

$$(V \otimes W)_p = \bigoplus_{i+j=p} V_i \otimes W_j$$

where $i, j, p \in G$. Hence, due to (8), $V \otimes W$ is a G-graded Jordan module of J with the action defined by

$$\{J, (V \otimes W)\} = JV \otimes W + V \otimes JW \tag{37}$$

and we may contract $V \otimes W$ by ϕ. Here, as in the case of Lie algebras in [2], the equality (37) is destroyed,

$$\{J, (V \otimes W)\}^\phi \neq (JV)^\phi \otimes W + V \otimes (JV)^\phi.$$

In order to restore (37) we introduce the tensor product contraction map τ:

$$(V_i \otimes W_j)^\tau = \tau_{ij} V_i \otimes W_j \tag{38}$$

in analogy with (18). In terms of grading subspaces (37) and (38) can be written as follows:

$$\{J_k, (V_i \otimes W_j)\} = J_k V_i \otimes W_j + V_i \otimes J_k W_j$$
$$\tau_{ij}\phi_{k,i+j}\{J_k, (V_i \otimes W_j)\} = \phi_{ki}\tau_{k+i,j}(J_k V_i)^{\phi} \otimes W_j + \phi_{kj}\tau_{i,k+j} V_i \otimes (J_k W_j)^{\phi}.$$

Both equations must hold simultaneously for all choices of elements of J_k, V_i and W_j. That is possible only in the case when for each fixed ϕ the following system of equations is satisfied:

$$\tau_{ij}\phi_{k,i+j} = \phi_{ki}\tau_{k+i,j} = \phi_{kj}\tau_{i,k+j}. \tag{39}$$

In order to find all tensor product contractions one needs to solve (39) for all fixed ϕ. Since we have obtained again the same equations for τ as for the tensor product contractions for the case of Lie algebras, a list of some solutions of (39) for specific gradings could be found in [2].

One also may note that in the case of fixed contraction matrix ε equation (39) is always solved by $\tau = \phi = \varepsilon$.

Acknowledgments

We are grateful to I Kaplansky, M Racine, I Shestakov and V Sokolov for helpful discussions and comments. We also express our gratitude to Yves Marcoz for computational work. One of us (IK) acknowledges the hospitality extended to her at the Centre de recherches mathématiques, Université de Montréal, where most of the work was done. One of the authors (IK) was supported by Fundação de Amparo á Pesquisa do Estado de São Paulo (N99/12223-5). The work was partially supported by the Natural Science and Engineering Research Council of Canada, by FQRC of Quebec and by NATO.

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